PMM U.S.S.R., Vol.51, No.2, pp.167-172, 1987
Printed in Great Britain

0021-8928/87 $10.00+0.00

ON A COMPUTATIONAL ALGORITHM FOR SOLVING GAME CONTROL PROBLEMS*

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A positional differential game of approach to a target is considered. The construction of the set of positional absorption (SPA) is studied. Relations are given, on the basis of which an algorithm of approximate computation of the SPA for controlled systems in the plane is developed.

Suppose we are given the control system whose behaviour in the time interval $[t_0, \vartheta]$ $(\vartheta > t_0)$ is described by the equation

$$dx/dt = f(t, x, u, v),\ x(t_0) = x_0;\ u \in P,\ v \in Q \tag{0.1}$$

Here, x is the m-dimensional phase vector of the system, which lies in Euclidean space R^m, u is the vector of control signals, v is a vector characterizing the noise acting on the system, and P, Q are compacta in Euclidean spaces R^p, R^q respectively.

The system is assumed to satisfy the standard conditions in the theory of differential games (/1/, p.32).

The task of constructing the SPA is allied to the task of constructing the positional strategy which guarantees that the target is hit at the instant ϑ /1-3/. The SPA is the set of all points for which such a strategy can be constructed. The SPA is known /1-6/ to be obtainable by path procedures, in which an absorption operator is used at each step. Questions of convergence in such procedures are also discussed in /7-10/.**(**See also, A.M. Taras'ev and V.N. Ushakov, On the construction of stable bridges in the min-max approach-evasion game, Sverdlovsk, Dep. at VINITI No.2454-83, 1983). In addition, problems concerned with computer modelling of SPA for linear systems are studied in /8, 9/.

Computational aspects of SPA construction are considered below. In Sect.1 the construction of the stable absorption operator is distinguished by means of a certain set of conditions, and this operator is the key to finding the SPA. The construction is a fairly general scheme which is suitable from the point of view of approximate computer evaluations. In Sect.2 the conditions are found under which a discrete approximation of SPA is convergent to it when the discretization step tends to zero. In Sect.3 relations are written, on the basis of which the algorithm of approximate calculation of the discrete approximation for controlled systems in the plane is developed. The realization of the algorithm is confined to cases when the elements of the discrete approximation are simply connected in phase space. Examples are quoted.

1. We introduce the concept of an operator of stable absorption and the u-stable bridge, and consider some versions of u-stability and prove their equivalence.

Let $M \subset R^m$ be the target set in the approach problem at instant ϑ /1-3/. We assume that all the construction below (of stable bridges, motions, and neighbourhoods of the target M) lie in a fairly large compact domain $D \subset [t_0, \vartheta] \times R^m$. Let $G = \{f \in R^m : \|f\| \leqslant K < \infty\}$ be a sphere such that $F(t, x) = \mathrm{co}\,\{f = f(t, x, u, v) : u \in P,\ v \in Q\} \subset G$, where $\mathrm{co}\,\{f\}$ is the closed convex hull of set $\{f\}$, and $\|f\|$ is the norm of vector f in Euclidean space.

Suppose we are given a set Ψ of elements ψ, and a family of mappings $\{F_\psi : D \mapsto 2^{R^m}\}$, corresponding to the set Ψ and satisfying the following conditions.

A1. Given any $(t, x, \psi) \in D \times \Psi$, the set $F_\psi(t, x)$ is convex and closed, and satisfies the inclusion $F_\psi(t, x) \subset G$ and further, the mapping $F_\psi : D \mapsto 2^{Rm}$ is upper semicontinuous for any $\psi \in \Psi$.

A2. Given any $(t, x, l) \in D \times S$

$$\begin{gathered}\min_{\psi \in \Psi} \rho_F(l) = \xi(t, x, l), \quad F = F_\psi(t, x) \\ S = \{l \in R^m : \|l\| = 1\}, \quad \rho_F(l) = \max_{f \in F} \langle l, f\rangle \\ \xi(t, x, l) = \max_{u \in P} \min_{v \in Q} \langle l, f(t, x, u, v)\rangle\end{gathered} \tag{1.1}$$

($\langle l, f\rangle$ is the scalar product of vectors l and f).

*Prikl.Matem.Mekhan., 51, 2, 216-222, 1987

Taking $W^* \subset R^m$, we introduce the notation $X_\psi(t_*; t^*, W^*) = \{x_* \in R^m : W^* \cap X_\psi(t^*; t_*, x_*) \neq \varnothing\}$, $X_\psi(t^*; t_*, x_*)$ is the set in R^m of all points at which arrive at instant t^* the solutions $x(\cdot) = (x(t): t_* \leqslant t \leqslant t^*, x(t_*) = x_*)$ of differential inclusion $\dot{x} \in F_\psi(t, x)$

$$Z_\psi(t_*; t^*, W^*) = \bigcup_{x^* \in W^*} Z_\psi(t_*; t^*, x^*),\ Z_\psi(t_*; t^*, x^*)$$

is the set in R^m of all points at which arrive at instant τ_* the solutions $z(\cdot) = (z(\tau): \tau^* \leqslant \tau \leqslant \tau_*,\ z(\tau^*) = x^*)$ of differential inclusion $\dot{z} \in \Phi_\psi(\tau, z)$, where $\Phi_\psi(\tau, z) = -F_\psi(t, z)$, $t + \tau = \vartheta$, $t_* + \tau_* = \vartheta$, $t^* + \tau^* = \vartheta$.

We give the definition of the operator of stable absorption.

Definition 1. The operator of stable absorption $\pi(t_*; t^*, W^*)$ $(t_0 \leqslant t_* < t^* \leqslant \vartheta,\ W^* \subset R^m)$ is the mapping $\pi(t_*; t^*, \cdot) : 2^{R^m} \mapsto 2^{R^m}$, given by the relation

$$\pi(t_*; t^*, W^*) = \bigcap_{\psi \in \Psi} X_\psi(t_*; t^*, W^*) = \bigcap_{\psi \in \Psi} Z_\psi(t_*; t^*, W^*) \tag{1.2}$$

Let $M \subset R^m$ and $W \subset D$ be closed sets.

Definition 2. We call set W a u-stable bridge in the problem of approach to M at fixed instant ϑ, if

$$W(\vartheta) \subset M,\ W(t_*) \subset \pi(t_*; t^*, W(t^*)),\quad \forall t_*, t^*\ (t_0 \leqslant t_* < t^* \leqslant \vartheta)\ (W(t) = \{x \in R^m : (t, x) \in W\}) \tag{1.3}$$

Assume that $\{F_\psi : D \mapsto 2^{R^m}\}$, $\{F_{\psi^*} : D \mapsto 2^{R^m}\}$ are the families of mappings corresponding to sets Ψ and Ψ^* and satisfying conditions A1, A2. Each family induces its own operator $\pi(t_*; t^*, W^*)$ of stable absorption. It can be shown that a set W which is u-stable in the sense of one operator is u-stable in the sense of the other, i.e., the concept of u-stability is invariant with respect to families of mappings which satisfy conditions A1, A2. In this sense definition 2 is well-posed.

We take

$$G_l(t, x) = G \cap \{f \in R^m : \langle l, f\rangle \leqslant \xi(t, x, l)\}\ (l \in S)$$

$F_{v(\cdot)}(t, x) = \mathrm{co}\,\{f(t, x, u, v(u)) : u \in P\}\ (v(\cdot) \in V$, V is the set of all mappings $v(\cdot) : P \mapsto Q)$.

The families of mappings $\{G_l : D \mapsto 2^{R^m}\}$, $\{F_{v(\cdot)} : D \mapsto 2^{R^m}\}$, corresponding to sets S and V respectively, satisfy conditions A1, A2, and hence the familiar definitions of u-stability, /1, 2, 7, 11/ fit into the scheme of Definitions 1, 2. Hence our Definition 2 is equivalent to the definitions of /1, 2, 7, 11/.

We take system (0.1) with the right-hand side

$$f(t, x, u, v) = f^{(1)} + f^{(2)},\ f^{(1)} = f^{(1)}(t, x, u),\ f^{(2)} = f^{(2)}(t, x, v) \tag{1.4}$$

which satisfies the conditions of /1/, where we put

$$F^{(1)}(t, x) = \mathrm{co}\,\{f^{(1)} : u \in P\},\ F^{(2)}(t, x) = \mathrm{co}\,\{f^{(2)} : v \in Q\}$$

We assume that $F^{(2)}(t, x)$ is a convex polyhedron, expressible as $F^{(2)}(t, x) = \mathrm{co}\,\{f^{(\omega)}(t, x) : \omega = 1, \ldots, n\}$, where $f^{(\omega)}$ $(\omega = 1, \ldots n)$ are continuous functions of t, x.

We put $F_\omega(t, x) = F^{(1)}(t, x) + f^{(\omega)}(t, x)$ $(\omega = 1, \ldots, n)$.

The family of mappings $\{F_\omega : D \mapsto 2^{R^m}\}$, corresponding to the set $\Omega = \{\omega = 1, \ldots, n\}$, satisfies conditions A1, A2. Thus, for system (0.1) with right-hand side (1.4) we can give a definition of u-stability in terms of the family of mappings $\{F_\omega : D \mapsto 2^{R^m}\}$, i.e., the operator of stable absorption for system (0.1) with right-hand side (1.4) can be written as

$$\pi(t_*; t^*, W^*) = \bigcap_{\omega=1}^{n} X_\omega(t_*; t^*, W^*) = \bigcap_{\omega=1}^{n} Z_\omega(t_*; t^*, W^*) \tag{1.5}$$
$$t_0 \leqslant t_* < t^* \leqslant \vartheta,\quad W^* \subset R^m$$

2. Consider the discrete (in time) analogues of the scheme of u-stability given in Sect. 1. We assume that the family of mappings $\{F_\psi : D \mapsto 2^{R^m}\}$, corresponding to set Ψ, satisfies the following in addition to conditions A1, A2:

A3. There is a function $\omega^*(\delta)$ $(\omega^*(\delta) \downarrow 0,\ \delta \downarrow 0)$, such that, given any (t_*, x_*), (t^*, x^*) of D and $\psi \in \Psi$,

$$d(F_\psi(t^*, x^*),\ F_\psi(t_*, x_*)) \leqslant \omega^*(|t^* - t_*| + \|x^* - x_*\|) \tag{2.1}$$

A4. There is a number $\lambda \in [0, \infty)$ such that, for any (t, x_*), (t, x^*) of D and $\psi \in \Psi$,

$$d(F_\psi(t, x^*),\ F_\psi(t, x_*)) \leqslant \lambda \|x^* - x_*\| \tag{2.2}$$

($d(F^*, F_*)$ is the Hausdorff distance between F^* and F_*).

We give a definition of the approximating system of sets (ASS), aimed at approximate calculation of the maximum u-stable bridge. The concept of ASS arises when the continuous scheme is replaced by a discrete scheme, i.e., when we introduce the division $\Gamma = \{t_0, t_1, \ldots, t_N = \vartheta\}$ of the interval $[t_0, \vartheta]$ and replace the domain of attainability $X_\psi(t^*; t_*, x_*)$ appearing in Definition 2 by the linear approximations $X_\psi{}^a(t^*; t_*, x_*)$ $(t_* = t_i,\ t^* = t_{i+1})$.

Definition 3. The approximating operator of stable absorption $\pi^a(t_*; t^*, W^*)$ $(t_0 \leqslant t_* < t^* \leqslant \vartheta,\ W^* \subset R^m)$ is the mapping $\pi^a(t_*; t^*, W^*) : 2^{R^m} \mapsto 2^{R^m}$, given by the relations

$$
\begin{aligned}
&\pi^a(t_*; t^*, W^*) = \bigcap_{\psi\in\Psi} X_\psi{}^a(t_*; t^*, W^*) \\
&X_\psi{}^a(t_*; t^*, W^*) = \{x_* \in R^m : W^* \cap X_\psi{}^a(t^*; t_*, x_*) \neq \varnothing\} \\
&X_\psi{}^a(t^*; t_*, x_*) = x_* + (t^* - t_*) F_\psi(t_*, x_*)
\end{aligned}
\tag{2.3}
$$

Definition 4. The ASS $\{W^a(t_i) : t_i \in \Gamma\}$ is the system for which

$$W^a(t_N) = M_{\varepsilon_N}, \quad W^a(t_i) = \pi^a(t_i; t_{i+1}, W^a(t_{i+1}))$$

for $i = N-1, N-2, \ldots, 0$; the number ε_N is found from the recurrence relations $\varepsilon_i = \omega(\Delta_{i-1}) + (1 + \lambda\Delta_{i-1})\varepsilon_{i-1}$, $\varepsilon_0 = 0$; $i = 1, 2, \ldots, N$; $\Delta_i = t_{i+1} - t_i$, $t_i \in \Gamma$, $\omega(\Delta) = \Delta\omega^*((1 + K)\Delta)$.

We denote by Φ_ε the closed ε-neighbourhood of set Φ.

Let $\{\Gamma_n\}$ be a sequence of divisions $\Gamma_n = \{t_0, t_1, \ldots, t_{N(n)}\}$ of the interval $[t_0, \vartheta]$, whose diameter $\Delta^{(n)} = \max\limits_i (t_{i+1} - t_i)$ tends to zero as $n \to \infty$. We have in mind that each Γ_n has its own $t_0, t_1, \ldots, t_{N(n)}$

Denote the ASS corresponding to division Γ_n by $\{W^{a,n}(t_i) : t_i \in \Gamma_n\}$.

Definition 5. Set W° consists of all the points $(t_*, x_*) \in D_t$ for which there exists the sequence

$$\{(t_n, x_n) : t_n = t_n(t_*) \in [t_0, \vartheta],\ x_n \in W^{a,n}(t_n),\ \lim_{n\to\infty} x_n = x_*\}$$

($t_n(t_*)$ is the instant of division Γ_n nearest on the right to t_*).

Theorem 1. Let $\{F_\psi : D \mapsto 2^{R^m}\}$ be the family of mappings which satisfies conditions A1-A4. Then W° is the maximum u-stable bridge.

The theorem gives a constructive way of approximately obtaining set W° on the basis of retrograde procedures, formalized in Definition 4.

In Definition 4 we so to speak started from the right-hand side

$$\pi(t_*; t^*, W^*) = \bigcap_{\psi\in\Psi} X_\psi(t_*; t^*, W^*)$$

of Eq.(1.2). We shall now take into account the second part of this equation and construct a new retrograde procedure.

Put

$$
\begin{aligned}
&Z_\psi{}^b(t_*; t^*, x^*) = x^* - (t^* - t_*) F_\psi(t^*, x^*) \\
&Z_\psi{}^b(t_*; t^*, \Phi^*) = \bigcup_{x^*\in\Phi^*} Z_\psi{}^b(t_*; t^*; x^*), \quad \Phi^* \subset R^m
\end{aligned}
$$

We assume that the family of mappings $\{F_\psi : D \mapsto 2^{R^m}\}$ also satisfies the following:

A5. There are numbers K^*, r^*, δ^* of $(0, \infty)$ such that, for any $(t^*, x^*) \in D$, $t_* \in [t_0, t^*]$, $\psi \in \Psi$, $r \in (0, (t^* - t_*) r^*)$, $t^* - t_* \leqslant \delta^*$

$$
\begin{aligned}
&(Z_\psi{}^b(t_*; t^*, x^*))_r \subset Z_\psi{}^b(t_*; t^*, U_{K^*r}(x^*)) \\
&(U_{K^*r}(x^*) = \{x \in R^m : \| x - x^* \| \leqslant K^* r\})
\end{aligned}
$$

Notice that condition A5 is not excessively restrictive. It is satisfied e.g., by the family of mappings $\{F_{v(\cdot)} : D \mapsto 2^{R^m}\}$.

Definition 6. The approximating operator of stable absorption $\pi^b(t_*; t^*, W^*)$ $(t_0 \leqslant t_* < t^* \leqslant \vartheta,\ W^* \subset R^m)$, is the mapping $\pi^b(t_*; t^*, \cdot) : 2^{R^m} \mapsto 2^{R^m}$, given by the relation

$$\pi^b(t_*; t^*, W^*) = \bigcap_{\psi\in\Psi} Z_\psi{}^b(t_*; t^*, W^*) \tag{2.4}$$

Definition 7. The ASS $\{W^b(t_i) : t_i \in \Gamma\}$ is the system for which

$$W^b(t_N) = M_{\zeta_N}, \quad W^b(t_i) = \pi^b(t_i; t_{i+1}, W^b(t_{i+1})), \quad i = N-1, \ldots, 0$$

where ζ_N is found from the recurrence relations

$$\zeta_i = (1 + K^*)\omega(\Delta_{i-1}) + (1 + \lambda\Delta_{i-1})\zeta_{i-1}, \quad \zeta_0 = 0, \; i = 1, 2, \ldots, N$$

Theorem 2. Let $\{F_\psi : D \mapsto 2^{R^m}\}$ be a family of mappings which satisfies conditions A1-A5. Then W° is the set of all points $(t_*, x_*) \in D$ which can be written as $(t_*, x_*) = \lim (t_n, x_n)$, $n \to \infty$, where $t_n = t_n(t_*)$ is the instant of division Γ_n nearest on the right to t_*, and $x_n \in W^b(t_n)$.

3. Consider the problem of approximately constructing set W° for system (0.1) with right-hand side (1.4). Using Theorems 1 and 2, the problem will be solved as a problem of approximately constructing system $\{W^a(t_i) : t_i \in \Gamma_n\}$ or a system $\{W^b(t_i) : t_i \in \Gamma_n\}$.

In addition to the conditions imposed on the right-hand side of system (0.1), we assume that we have the representation $F^{(1)}(t, x) = \operatorname{co}\{f_{(\gamma)}(t, x) : \gamma = 1, \ldots, k\}$, that functions $f_{(\gamma)}(t, x)$, $f^{(\omega)}(t, x)$ $(\gamma = 1, \ldots, k; \omega = 1, \ldots, n)$ satisfy the inequalities $\|f_{(\gamma)}(t, x)\| \leqslant \varkappa(1 + \|x\|)$, $\|f^{(\omega)}(t, x)\| \leqslant \varkappa(1 + \|x\|)$, $\varkappa > 0$, $(t, x) \in [t_0, \vartheta] \times R^m$, and a Lipschitz condition in $D^* = \{(t, x) : t \in [t_0, \vartheta], \|x\| \leqslant d_0 \exp 2\varkappa(\vartheta - t_0)\}$.

Here, $d_0 = \max\{2, d(M_\varepsilon, \{0\})\}$, $\varepsilon \in (0, \infty)$ is a fixed number.

Let division Γ have diameter $\Delta(\Gamma)$, satisfying the inequality $L(D^*)\Delta(\Gamma) \leqslant 1/2$, $1 - 2\varkappa\Delta(\Gamma) \geqslant 1/2$. We also assume that, with any $(t, x) \in D^*$, the set $\{f_{(\gamma)}(t, x) : \gamma = 1, \ldots, k\}$ is a $K^\circ\Delta(\Gamma)$-mesh on the set $\partial F^{(1)}(t, x)$, where $K^\circ \in (0, \infty)$ is independent of the choice of (t, x).

We introduce the notation

$$W^a(t_N) = M_\varepsilon, \quad X^a_{\omega,\gamma}(t_{i+1}; t_i, x(t_i)) = x(t_i) + \Delta_i(f^{(\omega)}(t_i, x(t_i)) + f_{(\gamma)}(t_i, x(t_i))), \; X^a_{\omega,\gamma}(t_i; t_{i+1}\, W^a(t_{i+1})) = \{x(t_i) \in R^m : X^a_{\omega,\gamma}(t_{i+1}; t_i, x(t_i)) \cap W^a(t_{i+1}) \neq \varnothing\}$$

We then have the approximate equation

$$X_\omega{}^a(t_i; t_{i+1}, W^a(t_{i+1})) \approx \bigcup_{\gamma=1}^{k} X^a_{\omega,\gamma}(t_i; t_{i+1}, W^a(t_{i+1})) \tag{3.1}$$

understood in the sense that

$$d(X_\omega{}^a(t_i; t_{i+1}, W^a(t_{i+1})) \bigcup_{\gamma=1}^{k} X^a_{\omega,\gamma}(t_i; t_{i+1}, W^a(t_{i+1}))) \leqslant 2K^c\Delta_i\Delta(\Gamma)$$

Hence follows the approximate equation

$$W^a(t_i) \approx \bigcap_{\omega=1}^{n} \bigcup_{\gamma=1}^{k} X^a_{\omega,\gamma}(t_i; t_{i+1}, W^a(t_{i+1})) \tag{3.2}$$

Using this, we replace the problem of constructing the system of sets $\{W^a(t_i) : t_i \in \Gamma\}$ by the problem of constructing system $\{W^{da}(t_i) : t_i \in \Gamma\}$, where sets $W^{da}(t_i)$ $(i = 0, 1, \ldots, N)$ are given recurrently:

$$W^{da}(t_N) = M_\varepsilon, \quad W^{da}(t_i) = \bigcap_{\omega=1}^{n} \bigcup_{\gamma=1}^{k} X^a_{\omega,\gamma}(t_i; t_{i+1}, W^{da}(t_{i+1})), \quad i = N-1, N-2, \ldots, 0; \quad X^a_{\omega,\gamma}(t_i; t_{i+1}, W^{da}(t_{i+1})) = \{x(t_i) \in R^m : W^{da}(t_{i+1}) \cap X^a_{\omega,\gamma}(t_{i+1}; t_i, x(t_i)) \neq \varnothing\}$$

The set $X^a_{\omega,\gamma}(t_i; t_{i+1}, W^{da}(t_{i+1}))$ can be written as

$$\bigcup_{x(t_{i+1}) \in W^{da}(t_{i+1})} x_{\omega,\gamma}(t_i; t_{i+1}, x(t_{i+1}))$$

where $x(t_i) = x_{\omega,\gamma}(t_i; t_{i+1}, x(t_{i+1}))$ is the solution of the equation

$$x + \Delta_i(f^{(\omega)}(t_i, x) + f_{(\gamma)}(t_i, x)) = x(t_{i+1}) \tag{3.3}$$

Notice that, in view of the conditions on the right-hand side (1.4) of system (0.1), and the choice of domain D^*, the solution $x = x(t_i)$ of Eq. (3.3) with $(t_{i+1}, x(t_{i+1})) \in W^{da}(t_{i+1}) \subset D^*$ is unique, and can be found as the limit $x(t_i) = \lim x^{(\rho)}(t_i)$, $x^{(\rho)}(t_i) = x^{(\rho)}_{\omega,\gamma}(t_i; t_{i+1}, x(t_{i+1}))$ as $\rho \to \infty$, where

$$x^{(0)}_{\omega,\gamma}(t_i; t_{i+1}, x(t_{i+1})) = x(t_{i+1})$$
$$x^{(\rho)}_{\omega,\gamma}(t_i; t_{i+1}, x(t_{i+1})) = x(t_{i+1}) - \Delta_i (f^{(\omega)}(t_i, x^{(\rho-1)}(t_i)) + f_{(\gamma)}(t_i, x^{(\rho-1)}(t_i))), \quad \rho = 1, 2, \ldots$$

Here we have the inclusions $(t_i, x(t_i)) \in D^*$, $(t_i, x^{(\rho)}(t_i)) \in D^*$ for $\rho = 1, 2, \ldots$.
We put

$$W^{da,\rho}(t_N) = M_\varepsilon, \quad W^{da,\rho}(t_i) = \bigcap_{\omega=1}^{n} \bigcup_{\gamma=1}^{k} X^{a,\rho}_{\omega,\gamma}(t_i; t_{i+1}, W^{da,\rho}(t_{i+1}))$$
$$(i = N-1, \ldots, 0)$$
$$X^{a,\rho}_{\omega,\gamma}(t_i; t_{i+1}, W^{da,\rho}(t_{i+1})) = \bigcup_{x(t_{i+1}) \in W^{da,\rho}(t_{i+1})} x^{(\rho)}_{\omega,\gamma}(t_i; t_{i+1}, x(t_{i+1}))$$

The system $\{W^{da,\rho}(t_i) : t_i \in \Gamma\}$ is the approximation of system $\{W^a(t_i) : t_i \in \Gamma\}$, obtained by calculating system $\{W^{d\imath}(t_i) : t_i \in \Gamma\}$ as a result of operating only with the ρ-th approximations of Eq.(3.3). Here, $(t_i, W^{da,\rho}(t_i)) \subset D^*$ for any $i = 0, 1, \ldots, N$.

A construction similar to the above can be given for the approximate calculation of system $\{W^b(t_i) : t_i \in \Gamma\}$.

We will consider an example illustrating the theory. Let the controlled system be described in the time interval $[0, 2]$ by the equations

$$x_1^{\cdot} = x_2 + v, \quad x_2^{\cdot} = -x_2 \cos(\pi + x_1^3) + x_1^3 + x_1^2 x_2 + u$$

where the scalar controls u and v are subject to the constraints $u \in [-0.5; 0.5]$, $v \in [-1, 0]$. The target M in the approach problem is the unit circle in the plane.

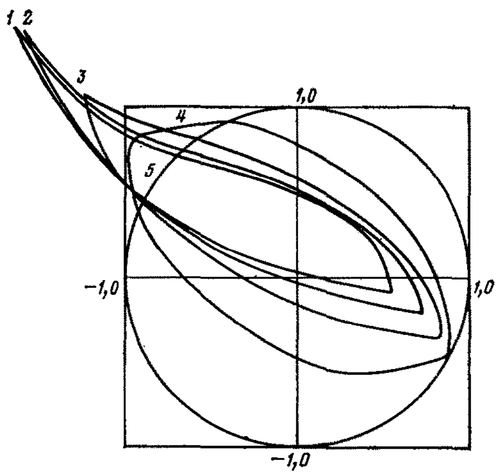

Fig.1

We computed the system of sets $\{W^{da,2}(t_i): t_i \in \Gamma\}$, corresponding to the division $\Gamma = \{t_0, t_1, \ldots, t_N\}$, where $t_0 = 0$, $t_N = \vartheta = 2$, $t_{i+1} = t_i + \Delta$, $\Delta = 0.05$. In Fig.1 we show the sets $W^{da,2}(t_i)$, corresponding to instants 0.0, 0.5, 1.0, 1.5, 2.0. Each of the five sets is a compactum, bounded by a curve marked by the appropriate number in Fig.1.

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Translated by D.E.B.

PMM U.S.S.R.,Vol.51,No.2,pp.172-177,1987
Printed in Great Britain

0021-8928/87 $10.00+0.00

REDUCING THE EQUATIONS OF MOTION OF CERTAIN NON-HOLONOMIC CHAPLYGIN SYSTEMS TO LAGRANGIAN AND HAMILTONIAN FORM*

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Non-holonomic Chaplygin systems /1/ with n degrees of freedom and $m\ (m < n)$ first integrals linear with respect to velocities, are considered. It is assumed that Lagrange's function is constructed taking into account the non-holonomic constraints imposed on the system, and the integrals are independent of the first m generalized coordinates. Then, provided that certain conditions are met, m linear non-holonomic coordinates (quasi-coordinates) can be introduced in such a way that the first m equations of motion in these coordinates will have the form of the usual Lagrange's equations.

The present paper deals with the most interesting, integrable case, when $m = n - 1$. It is shown that if certain conditions are met, the trajectories of such a system in phase space will represent quasiperiodic windings on the n-dimensional tori. Examples are given, namely, of a solid of revolution rolling along a fixed horizontal plane, and of the motion of a circular disc with a sharp edge on a smooth, horizontal ice surface.

The problem of reducing Chaplygin's equations of motion of non-holonomic systems to the form of the ordinary Lagrangian and Hamiltonian equations has been studied extensively. A detailed survey and an analysis of the existing approaches to solving this problem are given in /2/.

1. Let us consider a natural, non-holonomic mechanical Chaplygin system /1/ acted upon by potential forces. We assume that Lagrange's function constructed taking into account the non-integrable constraints imposed on the system, has the form

$$L(\mathbf{q}, \mathbf{q}^{\cdot}) = T - \Pi, \quad T = {}^1/_2\mathbf{q}^{\cdot T}\boldsymbol{\Omega}\mathbf{q}^{\cdot}, \quad \Pi = \Pi(\mathbf{q}) \tag{1.1}$$
$$\boldsymbol{\Omega} = \|\omega_{ij}(\mathbf{q})\| \quad (i, j = 1, 2, \ldots, n)$$

Here $\mathbf{q}$, $\mathbf{q}^{\cdot}$ are column matrices of the generalized coordinates and velocities of the system, $\boldsymbol{\Omega}$ is a positive definite symmetric $n \times n$-matrix, T and Π is the kinetic and potential energy of the system respectively. The total energy of the system is conserved ($T + \Pi = h = \text{const}$), and the differential Chaplygin equations of motion

$$\frac{d}{dt}\frac{\partial L}{\partial \mathbf{q}^{\cdot}} - \frac{\partial L}{\partial \mathbf{q}} = \boldsymbol{\Gamma} \tag{1.2}$$

will describe the motion of the system independently of the equations of non-integrable constraints. In (1.2) $\boldsymbol{\Gamma}$ is a column matrix of the non-holonomic terms ($\Gamma_i(\mathbf{q}, \mathbf{q}^{\cdot})$ is the quadratic

*Prikl.Matem.Mekhan.,51,2,223-229,1987